The List Polynomial

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Work done under the supervision of
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Thinking about lists…

$L(X)$: finite lists on a set X

Consider: the set of lists on X of length 3.
This is just $X \times X \times X$.

The set of lists of length 3 on $X = \mathbb{Z}$

$$\left\{ \begin{array}{c} [1, 5, -1], \\ [3, -2, 9], \\ [0, 0, 12], \\ [-4, -1, 1], \\ \dots \end{array} \right\}$$

Thinking about lists…

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This is just $X \times X \times X$.

Therefore: $L(X) = \sum_{n \in \mathbb{N}} X^n$

The set of lists of length 3 on $X = \mathbb{Z}$

$$\left\{ \begin{matrix} [1, 5, -1], \\ [3, -2, 9], \\ [0, 0, 12], \\ [-4, -1, 1], \\ \dots \end{matrix} \right\}$$

It’s a polynomial!

$X \mapsto L(X)$ is a functor **Set** → **Set**

It has a particular shape: $L(X) = \sum_{n \in \mathbb{N}} X^n$

It is a ***polynomial functor***.

Let’s generalize!

Usual polynomials:

$$1 + x^2 + x^7$$
$$=$$
$$\sum_{i \in I} x^{a_i}$$

for $I = \{1,2,3\}$,
$a_1 = 0, a_2 = 2, a_3 = 7.$

$L(X)$: lists on a set X

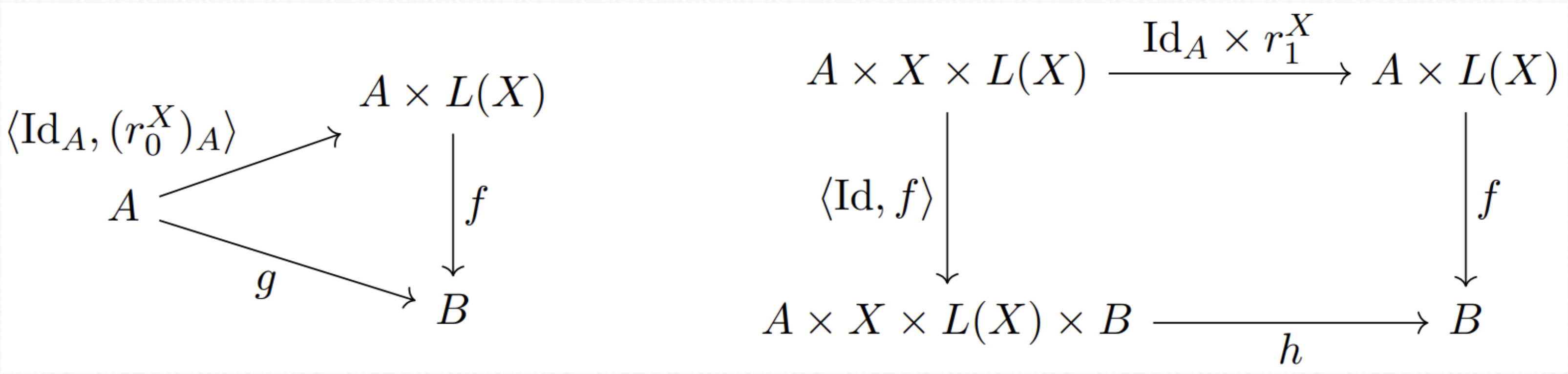
List objects in general

In a category with finite products:

- List object $L(X)$
- Empty list $1 \to L(X)$
- Append operation $X \times L(X) \to L(X)$

Inductively define $f : A \times L(X) \to B$

- $f(a, \emptyset) = g(a)$
- $f(a, x :: \ell) = h\big(a, x, \ell, f(a)\big)$



List objects in general

If a category $\mathcal{C}$ has finite products, and for each object X there is a list object $L(X)$, then we can form a functor:

$$L : \mathcal{C} \to \mathcal{C} : X \mapsto L(X)$$

Is this a polynomial functor?

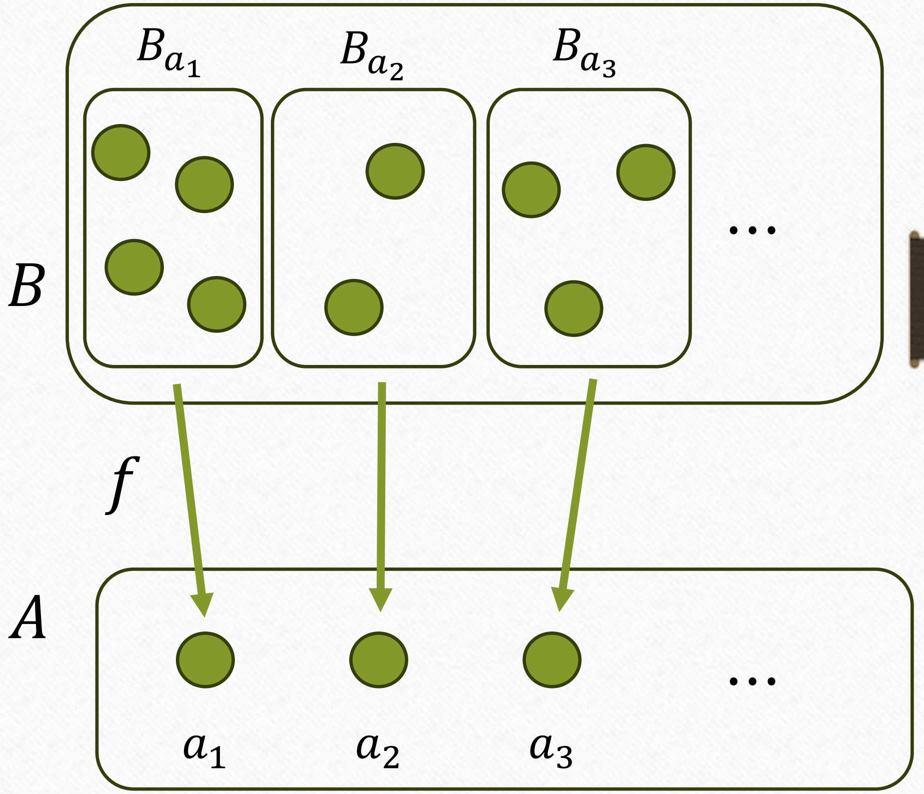
What is a polynomial functor?

Internal language of slice categories

In a slice category $\mathcal{C}/A$, an object $f : B \to A$ is thought of as a collection of sets indexed by A:

$$(B_a \mid a \in A)$$

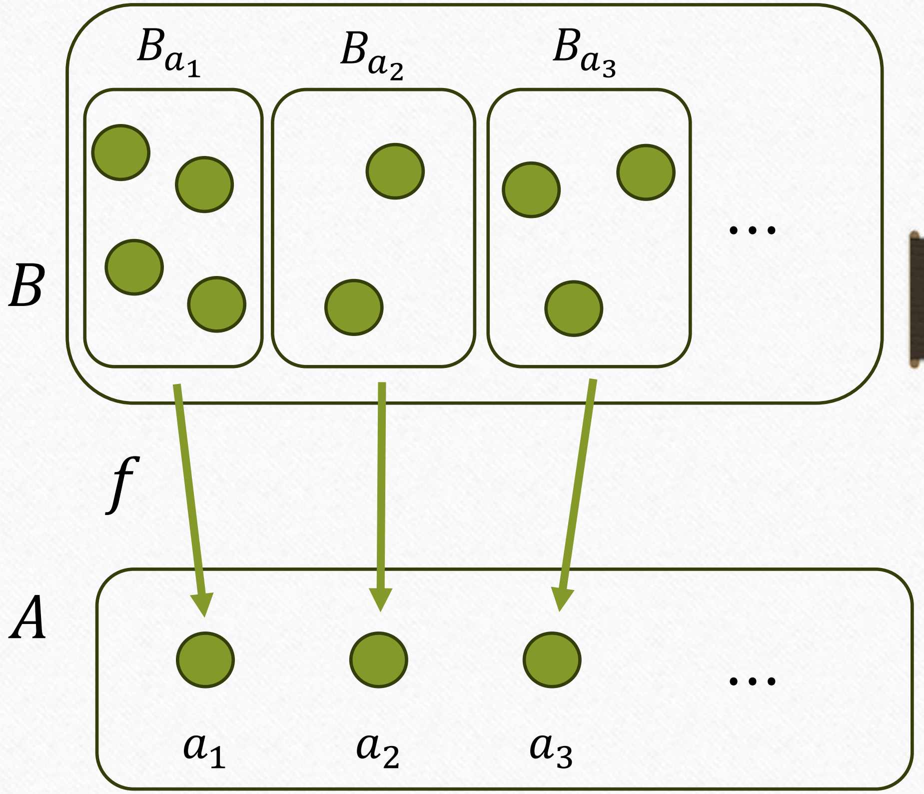
" $B_a = f^{-1}(a)$ " for " $a \in A$ "



Adding syntax

If $f : B \to A$ is denoted $(B_a \mid a \in A)$, how should we interpret the following notation?

$$\sum_{a \in A} B_a$$



In the slice category $\mathcal{C}/A$

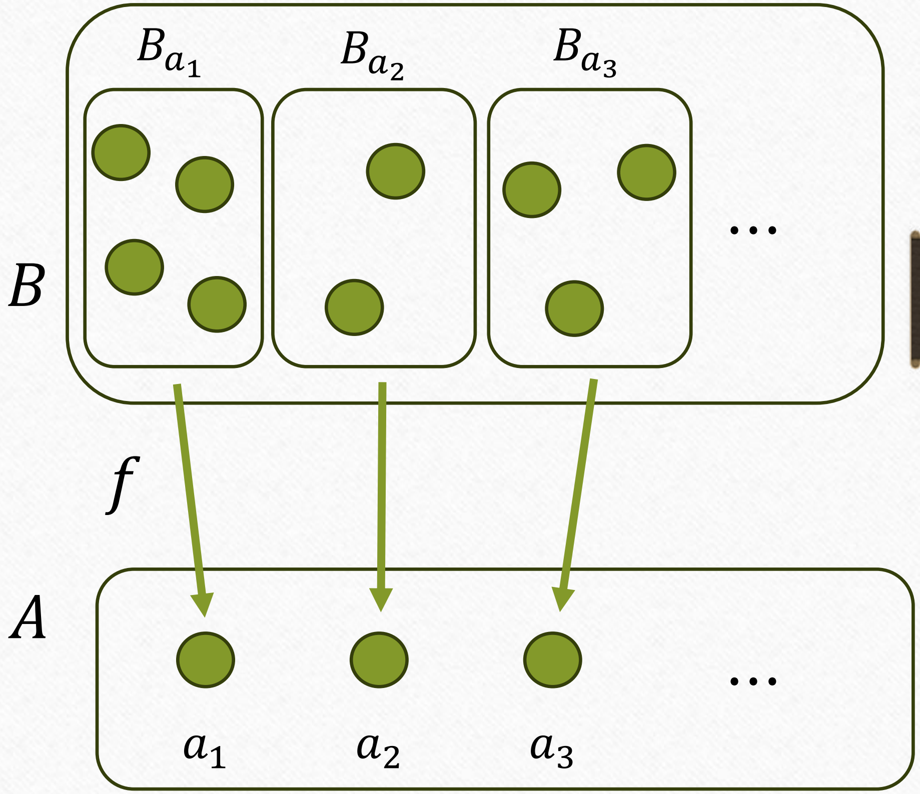
Adding syntax

If $f : B \to A$ is denoted $(B_a \mid a \in A)$, how should we interpret the following notation?

$$\sum_{a \in A} B_a = B$$

There is a functor $\Sigma_A : \mathcal{C}/A \to \mathcal{C}$ given by $(f : B \to A) \mapsto B$

Or: $(B_a \mid a \in A) \mapsto \Sigma_{a \in A} B_a$.



In the slice category $\mathcal{C}/A$

Adding syntax

How should we interpret the following notations? What is the corresponding $f : B \to A$?

- $(1 \mid a \in A)$
- $(X \mid a \in A)$
- $(C_a \times D_a \mid a \in A)$

In the slice category $\mathcal{C}/A$

Adding syntax

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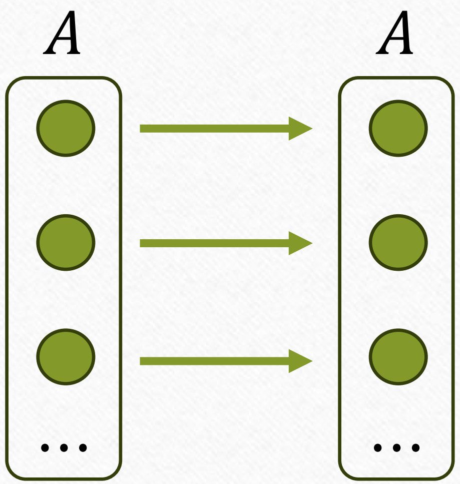
- $(1 \mid a \in A)$
- $(X \mid a \in A)$
- $(C_a \times D_a \mid a \in A)$

- $Id : A \to A$

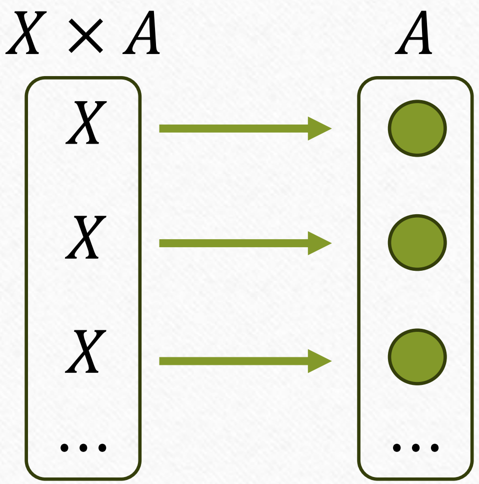


In the slice category $\mathcal{C}/A$

Adding syntax

How should we interpret the following notations? What is the corresponding $f : B \to A$?

- $(1 \mid a \in A)$
- $Id : A \to A$
- $(X \mid a \in A)$
- $\pi_2 : X \times A \to A$
- $(C_a \times D_a \mid a \in A)$



In the slice category $\mathcal{C}/A$

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- $(X \mid a \in A)$
- $(C_a \times D_a \mid a \in A)$
- $Id : A \to A$
- $\pi_2 : X \times A \to A$
- $C \times_A D \to A$

In the slice category $\mathcal{C}/A$

Slices of categories with finite limits

If $\mathcal{C}$ has finite limits, then so does $\mathcal{C}/A$.

There is a functor $\Delta_A : \mathcal{C} \to \mathcal{C}/A$ given by $X \mapsto (\pi_2 : X \times A \to A)$.

Or: $X \mapsto (\, X \mid a \in A \,)$.

Adding syntax

If $g : C \to A$ and $f : B \to A$ are represented by $(C_a \mid a \in A)$ and $(B_a \mid a \in A)$, respectively, how should we interpret the following notation?

$$\left((C_a)^{B_a} \mid a \in A \right)$$

In the slice $\mathcal{C}/A$ of a category $\mathcal{C}$ with finite limits

Adding syntax

If $g : C \to A$ and $f : B \to A$ are represented by $(C_a \mid a \in A)$ and $(B_a \mid a \in A)$, respectively, how should we interpret the following notation?

$$\left((C_a)^{B_a} \mid a \in A\right)$$

The exponential of g and f in $\mathcal{C}/A$ (if it exists!)

Recall: f is **exponentiable** (in $\mathcal{C}/A$) if the exponential g^f exists for any g. In this case, we have a functor $(-)^f : \mathcal{C}/A \to \mathcal{C}/A$.

In the slice $\mathcal{C}/A$ of a category $\mathcal{C}$ with finite limits

Polynomial functors, finally!

Let $f : B \to A$ be an arrow in a category $\mathcal{C}$ with finite limits. Assume f is exponentiable in $\mathcal{C}/A$.

Write f in the internal language of $\mathcal{C}/A$ as $(\, B_a \mid a \in A \,)$. The **polynomial functor $\boldsymbol{P_f}$ associated to $\boldsymbol{f}$** is:

$$\mathcal{C} \xrightarrow{\Delta_A} \mathcal{C}/A \xrightarrow{(-)^f} \mathcal{C}/A \xrightarrow{\Sigma_A} \mathcal{C}$$

$$X \longmapsto (\, X \mid a \in A \,) \longmapsto (\, X^{B_a} \mid a \in A \,) \longmapsto \sum_{a \in A} X^{B_a}$$

The list polynomial?

A functor $P : \mathcal{C} \to \mathcal{C}$ is **polynomial** if $P \cong P_f$ for some $f : B \to A$ (which must be exponentiable in $\mathcal{C}/A$).

So… is $L : X \to L(X)$ polynomial?

P_f: the polynomial functor associated to f
$L(X)$: list object on an object X

The list polynomial?

We claim $L(X) = \sum_{n \in \mathbb{N}} X^n$. That means $L : X \mapsto L(X)$ is isomorphic to the polynomial functor associated to the arrow $(n \mid n \in \mathbb{N})$ in $\mathcal{C}/\mathbb{N}$.

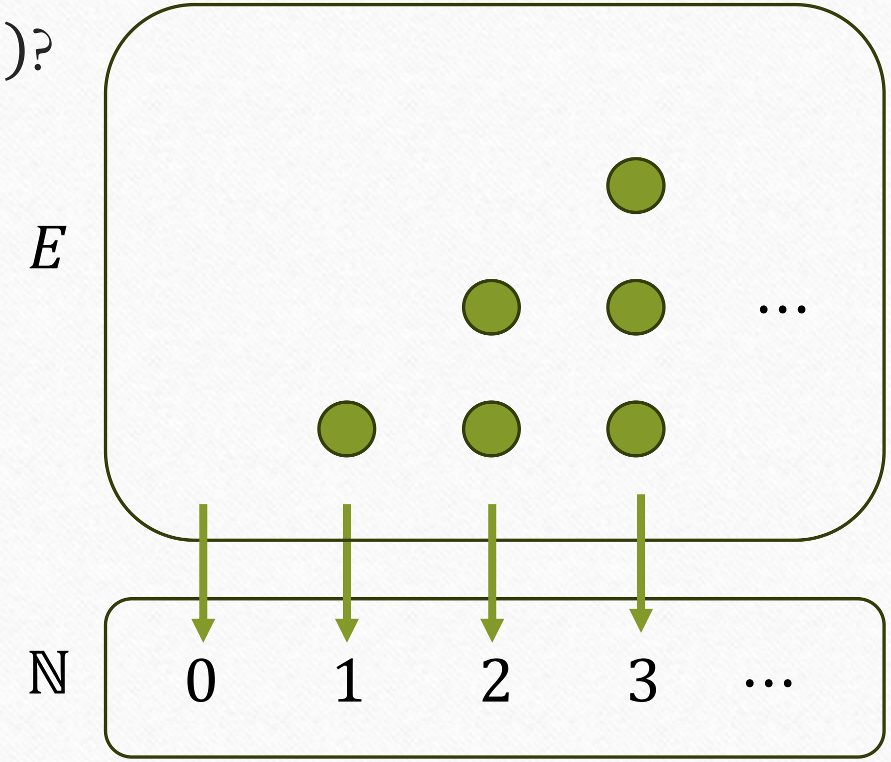
- What is $\mathbb{N}$?
- What arrow $E \to \mathbb{N}$ corresponds to $(n \mid n \in \mathbb{N})$?

$L(X)$: list object on an object X

The polynomial functor associated to $(B_a \mid a \in A)$ is $X \mapsto \sum_a X^{B_a}$

The list polynomial?

What arrow $E \to \mathbb{N}$ corresponds to $(n \mid n \in \mathbb{N})$?

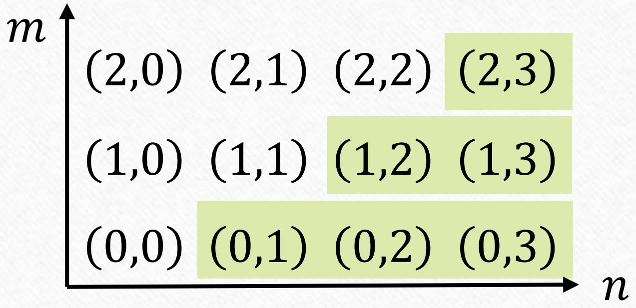


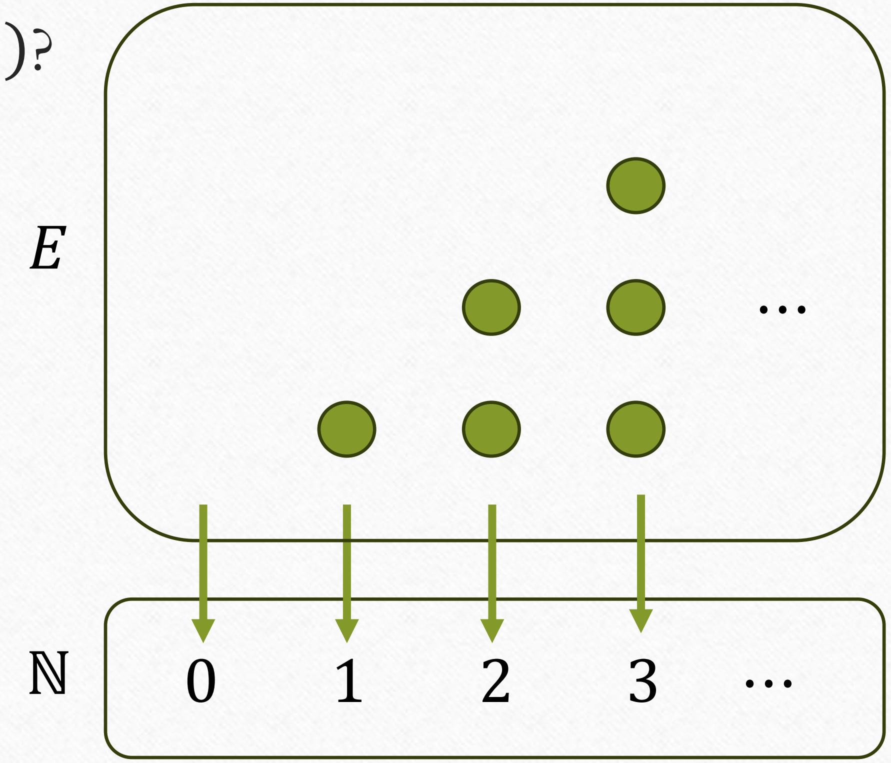
ℕ: an object which acts like the natural numbers

The list polynomial?

What arrow $E \to \mathbb{N}$ corresponds to $(n \mid n \in \mathbb{N})$?

$$E = \{(m, n) : m < n\}$$
$$\pi_2^E : E \to \mathbb{N}$$





$\mathbb{N}$: an object which acts like the natural numbers

The list polynomial (supposedly)

We claim that $\pi_2^E : E \to \mathbb{N}$ is exponentiable, and that the functor $L : X \mapsto L(X)$ is isomorphic to:

$$\begin{array}{ccccccc} \mathcal{C} & \xrightarrow{\Delta_{\mathbb{N}}} & \mathcal{C}/\mathbb{N} & \xrightarrow{(-)^{\pi_2^E}} & \mathcal{C}/\mathbb{N} & \xrightarrow{\Sigma_{\mathbb{N}}} & \mathcal{C} \\ X & \longmapsto & (X \mid n \in \mathbb{N}) & \longmapsto & (X^n \mid n \in \mathbb{N}) & \longmapsto & \sum_{n \in \mathbb{N}} X^n \end{array}$$

How do we prove this?

$L(X)$: list object on an object X

$\mathbb{N}$: an object which acts like the natural numbers

$\pi_2^E : E \to \mathbb{N}$ is an arrow which intuitively satisfies $(\pi_2^E)^{-1}(n) = \{1,2, \dots, n\}$

The right adjoints

$$length(\emptyset) = 0,$$
$$length(x :: \ell) = length(\ell) + 1$$

Consider the decomposition $L = \Sigma_{\mathbb{N}} \circ L_{\mathbb{N}}$:

$$\mathcal{C} \xrightarrow{L_{\mathbb{N}} \,:\, X \mapsto (length \,:\, L(X) \to \mathbb{N})} \mathcal{C}/\mathbb{N} \xrightarrow{\Sigma_{\mathbb{N}}} \mathcal{C}$$

$$\mathcal{C} \underset{\Sigma_{\mathbb{N}}}{\overset{\Delta_{\mathbb{N}}}{\rightleftarrows}} \mathcal{C}/\mathbb{N} \underset{\pi_2^E \times (-)}{\overset{(-)^{\pi_2^E}}{\rightleftarrows}} \mathcal{C}/\mathbb{N} \xrightarrow{\Sigma_{\mathbb{N}}} \mathcal{C}$$

$L(X)$: list object on an object X

$\mathbb{N}$: an object which acts like the natural numbers

$\Sigma_{\mathbb{N}} : (A \to \mathbb{N}) \mapsto A, \quad \Delta_{\mathbb{N}} : X \mapsto (\pi_2 : X \times \mathbb{N} \to \mathbb{N})$

In short:

To prove the list object functor is polynomial, we just need to show that

$$L_{\mathbb{N}} : \mathcal{C} \to \mathcal{C}/\mathbb{N}$$
$$X \mapsto (length : L(X) \to \mathbb{N})$$

is a right adjoint to

$$\mathcal{C} \xleftarrow{\;\Sigma_{\mathbb{N}}\;} \mathcal{C}/\mathbb{N} \xleftarrow{\;\pi_2^E \times (-)\;} \mathcal{C}/\mathbb{N}$$

$$E \times_{\mathbb{N}} A \longleftarrow\!\shortmid (A \to \mathbb{N})$$

Basic strategy

We construct a natural transformation $(\varepsilon_X : E \times_{\mathbb{N}} L(X) \to X)_X$ (which will be the co-unit) and show it satisfies the appropriate universal property:

For every $l_A : A \to \mathbb{N}$ and every $g : E \times_{\mathbb{N}} A \to X$,
there exists a unique $h : A \to L(X)$ such that
$length \circ h = l_A$ and $\varepsilon_X \circ (Id \times_{\mathbb{N}} h) = g$.

Requirements

We need to make the following assumptions about the category $\mathcal{C}$:

- $\mathcal{C}$ has all finite limits and list objects

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We need to make the following assumptions about the category $\mathcal{C}$:

- $\mathcal{C}$ has all finite limits and list objects
- $\mathcal{C}$ is **extensive**, though we can weaken this assumption to the following:

$$
\begin{array}{ccccc}
A_0 & \longrightarrow & A & \longleftarrow & A_{>0} \\
\downarrow & & \downarrow{\scriptstyle l_A} & & \downarrow \\
\mathbb{1} & \xrightarrow{\ 0\ } & N & \xleftarrow{\ s\ } & N
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{1} & \xrightarrow{r_0^X} & L(X) & \xleftarrow{r_1^X} & X \times L(X) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{len}_X} & & \downarrow{\scriptstyle \mathrm{len}_X \circ \pi_2} \\
\mathbb{1} & \xrightarrow{\ 0\ } & N & \xleftarrow{\ s\ } & N
\end{array}
$$

In an **extensive** category with finite limits and list objects, the list object functor is polynomial

In an **extensive** category with finite limits and list objects, the list object functor is polynomial

Thanks for listening!